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Some generalization of Cauchy's and Wilson's functional equations on abelian groups

RADOSŁAW ŁUKASIK

Abstract. We find the solutions $f, g, h: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}$ of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + \alpha(x)h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G , X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

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1. Introduction

The following generalization

$$f(x + y) + f(x + \sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

of the quadratic functional equation, where σ is an automorphism of an abelian group G such that $\sigma \circ \sigma = id_G$ and $f: G \rightarrow \mathbb{C}$, was investigated by Stetkær [9].

In his other work [10] he solved the functional equation

$$\frac{1}{N} \sum_{n=0}^{N-1} f(z + \omega^n \zeta) = f(z) + g(z)h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \dots\}$, ω is a primitive N^{th} root of unity, $f, g, h: \mathbb{C} \rightarrow \mathbb{C}$ are continuous.

Łukasik [5, 6] derived explicit formulas for the solutions of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|\alpha(y)g(x) + |K|h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite abelian subgroup of the automorphism group of G , X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $f, g, h: G \rightarrow X$, $\alpha: G \rightarrow \mathbb{K}$.

The functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x)h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite subgroup of the automorphism group on G , $f, g, h: G \rightarrow \mathbb{C}$, was studied by Förg-Rob and Schwaiger [3], Gajda [4], Stetkær [7, 8], Badora [2].

Aczél et al. [1] studied the complex-valued solutions of the equation

$$f(x + y) + f(x - y) - 2f(x) = g(x)h(y), \quad x, y \in G,$$

where $(G, +)$ is a group and $f, g, h: G \rightarrow \mathbb{C}$.

The purpose of this paper is to find the solutions of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + \alpha(x)h(y), \quad x, y \in G,$$

where $f, g, h: G \rightarrow X$, $\alpha: G \rightarrow \mathbb{K}$, $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G , X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

We find these solutions under the assumption $\sum_{\lambda \in K} f \circ \lambda \neq \text{const}$, $\sum_{\lambda \in K} \alpha \circ \lambda \neq \text{const}$, $\alpha(0) \neq 0$ and they are some combinations of multiplicative and multi-additive functions.

Our results generalize all the results mentioned above (except the papers by Łukasik).

2. Main result

Throughout the present paper, we assume that X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G .

In this work we use some theorems. The first gives us the form of the solutions of a generalization of Jensen's functional equation.

Theorem 1. [5, Theorem 5] *Let $(S, +)$ be an abelian semigroup, K be a finite subgroup of the automorphism group of S , $(H, +)$ be an abelian group uniquely divisible by $|K|!$. Then the function $f: S \rightarrow H$ satisfies the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|f(x), \quad x, y \in S \tag{1}$$

if and only if there exist k -additive, symmetric mappings $A_k: S^k \rightarrow H$, $k \in \{1, \dots, |K| - 1\}$ and $A_0 \in H$ such that

$$f(x) = A_0 + A_1(x) + \dots + A_{|K|-1}(x, \dots, x), \quad x \in S,$$

$$\sum_{\lambda \in K} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k \leq |K| - 1.$$

The second theorem shows all solutions of a generalization of Wilson's functional equation.

Theorem 2. [6, Theorem 4,5] Let $f: G \rightarrow X, f \neq 0, \varphi: G \rightarrow \mathbb{K}$. They satisfy the equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| \varphi(y) f(x), \quad x, y \in G, \quad (2)$$

if and only if there exists a homomorphism $m: G \rightarrow \mathbb{C}^*$, such that

$$\varphi(x) = \frac{1}{|K|} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and

- (i) if X is complex, then there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right], \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, \quad 1 \leq i \leq k \leq |K_0| - 1,$$

- (ii) if X is real, then there exist $A_0^\lambda \in X, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re}(m(\lambda x)) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] \right.$$

$$\left. - \operatorname{Im}(m(\lambda x)) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, \quad 1 \leq i \leq k \leq |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, \quad 1 \leq i \leq k \leq |K_0| - 1,$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, K_1 is the set of representatives of cosets of the quotient group K/K_0 .

First we start with a corollary of Theorem 2.

Corollary 1. *A nonzero function $\alpha: G \rightarrow \mathbb{K}$ satisfies the equation*

$$\sum_{\lambda \in K} \alpha(x + \lambda y) = \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y), \quad x, y \in G, \quad (3)$$

if and only if there exists a homomorphism $m: G \rightarrow \mathbb{C}^$ and*

- (i) *if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k -additive, symmetric mappings $a_k^\lambda: G^k \rightarrow \mathbb{C}, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that*

$$\begin{aligned} \alpha(x) &= \sum_{\lambda \in K_1} m(\lambda x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right], \quad x \in G, \\ \sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\lambda \in K_1} a_0^\lambda &= 1, \end{aligned}$$

- (ii) *if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k -additive, symmetric mappings $a_k^\lambda, b_k^\lambda: G^k \rightarrow \mathbb{R}, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that*

$$\begin{aligned} \alpha(x) &= \sum_{\lambda \in K_1} \left(\operatorname{Re}(m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right] \right. \\ &\quad \left. - \operatorname{Im}(m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \dots, x) \right] \right), \quad x \in G, \\ \sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\mu \in K_0} b_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\lambda \in K_1} a_0^\lambda &= 1, \end{aligned}$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, K_1 is the set of representatives of co-sets of the quotient group K/K_0 .

Moreover, if α has the above form, then

$$\sum_{\lambda \in K} \alpha(\lambda x) = \sum_{\lambda \in K} m(\lambda x), \quad x \in G.$$

Proof. Assume that α satisfies (3). Let $\varphi: G \rightarrow \mathbb{K}$ be given by the formula

$$\varphi = \frac{1}{|K|} \sum_{\lambda \in K} \alpha \circ \lambda.$$

Then α and φ satisfy the equation

$$\frac{1}{|K|} \sum_{\lambda \in K} \alpha(x + \lambda y) = \varphi(y) \alpha(x), \quad x, y \in G.$$

In view of Theorem 2 we have the form of α such as in the statement of this corollary and the equality

$$\varphi(x) = \frac{1}{|K|} \sum_{\lambda \in K} m(\lambda x), \quad x \in G.$$

We observe that

(i) if $\mathbb{K} = \mathbb{C}$, then

$$\begin{aligned} |K|\varphi(x) &= \sum_{\mu \in K} \alpha(\mu x) = \sum_{\mu \in K} \sum_{\lambda \in K_1} m(\lambda \mu x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(\mu x, \dots, \mu x) \right] \\ &= \sum_{\mu \in K_1} \sum_{\lambda \in K_1} m(\lambda \mu x) \left[|K_0| a_0^\lambda + \sum_{i=1}^{|K_0|-1} \sum_{\sigma \in K_0} a_i^\lambda(\sigma \mu x, \dots, \sigma \mu x) \right] \\ &= \sum_{\mu \in K_1} \sum_{\lambda \in K_1} m(\lambda \mu x) |K_0| a_0^\lambda = \sum_{\mu \in K} m(\mu x) \sum_{\lambda \in K_1} a_0^\lambda, \quad x, y \in G, \end{aligned}$$

(ii) if $\mathbb{K} = \mathbb{R}$, then

$$\begin{aligned} |K|\varphi(x) &= \sum_{\mu \in K} \alpha(\mu x) = \sum_{\mu \in K} \sum_{\lambda \in K_1} \operatorname{Re}(m(\lambda \mu x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(\mu x, \dots, \mu x) \right] \\ &\quad - \sum_{\mu \in K} \sum_{\lambda \in K_1} \operatorname{Im}(m(\lambda \mu x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(\mu x, \dots, \mu x) \right] \\ &= \sum_{\mu \in K_1} \sum_{\lambda \in K_1} \operatorname{Re}(m(\lambda \mu x)) \left[|K_0| a_0^\lambda + \sum_{i=1}^{|K_0|-1} \sum_{\sigma \in K_0} a_i^\lambda(\sigma \mu x, \dots, \sigma \mu x) \right] \\ &\quad - \sum_{\mu \in K_1} \sum_{\lambda \in K_1} \operatorname{Im}(m(\lambda \mu x)) \left[|K_0| b_0^\lambda + \sum_{i=1}^{|K_0|-1} \sum_{\sigma \in K_0} b_i^\lambda(\sigma \mu x, \dots, \sigma \mu x) \right] \\ &= \sum_{\mu \in K_1} \sum_{\lambda \in K_1} \operatorname{Re}(m(\lambda \mu x)) |K_0| a_0^\lambda - \sum_{\mu \in K_1} \sum_{\lambda \in K_1} \operatorname{Im}(m(\lambda \mu x)) |K_0| b_0^\lambda \\ &= \sum_{\mu \in K} \operatorname{Re}(m(\mu x)) \sum_{\lambda \in K_1} a_0^\lambda - \sum_{\mu \in K} \operatorname{Im}(m(\mu x)) \sum_{\lambda \in K_1} b_0^\lambda \\ &= \sum_{\mu \in K} m(\mu x) \sum_{\lambda \in K_1} a_0^\lambda, \quad x, y \in G. \end{aligned}$$

Hence $\sum_{\lambda \in K_1} a_0^\lambda = 1$ and on the other hand a function α , which has the form such as in the statement of this corollary, satisfies Eq. (3). \square

Theorem 3. *Let functions $f: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}$ be such that $\sum_{\lambda \in K} f \circ \lambda \neq 0$, $\alpha \neq 0$, $\sum_{\lambda \in K} \alpha \circ \lambda \neq |K|$. They satisfy the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|f(x) + \alpha(x) \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G, \quad (4)$$

if and only if there exist a homomorphism $m: G \rightarrow \mathbb{C}^$, $A_0 \in X$, k -additive, symmetric mappings $A_k: G^k \rightarrow X, k \in \{1, \dots, L-1\}$ such that*

$$f(x) = A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \alpha(x)A_0, \quad x \in G, \quad (5)$$

$$\sum_{\mu \in K} A_k(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad 1 \leq i \leq k < |K|, \quad (6)$$

and

- (i) *if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k -additive, symmetric mappings $a_k^\lambda: G^k \rightarrow \mathbb{C}, k \in \{1, \dots, |K_0|-1\}, \lambda \in K_1$ such that*

$$\alpha(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right], \quad x \in G, \quad (7)$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (8)$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \quad (9)$$

- (ii) *if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k -additive, symmetric mappings $a_k^\lambda, b_k^\lambda: G^k \rightarrow \mathbb{R}, k \in \{1, \dots, |K_0|-1\}, \lambda \in K_1$ such that*

$$\alpha(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re}(m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right] \right. \quad (10)$$

$$\left. - \operatorname{Im}(m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \dots, x) \right] \right), \quad x \in G, \quad (11)$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (12)$$

$$\sum_{\lambda \in K_0} b_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (13)$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \quad (14)$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, K_1 is the set of representatives of cosets of the quotient group K/K_0 .

Moreover

$$\sum_{\lambda \in K} f(\lambda x) = \left(|K| - \sum_{\lambda \in K} \alpha(\lambda x) \right) A_0, \quad x \in G. \quad (15)$$

Proof. Let f and α satisfy (4). We observe that

$$\begin{aligned} \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda y + \mu z) &= |K| \sum_{\lambda \in K} f(x + \lambda y) \\ &+ \sum_{\lambda \in K} \alpha(x + \lambda y) \sum_{\mu \in K} f(\mu z) = |K|^2 f(x) + |K| \alpha(x) \sum_{\lambda \in K} f(\lambda y) \\ &+ \sum_{\lambda \in K} \alpha(x + \lambda y) \sum_{\mu \in K} f(\mu z), \quad x, y, z \in G, \end{aligned}$$

and

$$\begin{aligned} \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda y + \mu z) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda(y + \mu z)) \\ &= |K|^2 f(x) + \alpha(x) \sum_{\mu \in K} \sum_{\lambda \in K} f(\lambda y + \mu z) \\ &= |K|^2 f(x) + |K| \alpha(x) \sum_{\lambda \in K} f(\lambda y) + \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y) \sum_{\mu \in K} f(\mu z), \quad x, y, z \in G. \end{aligned}$$

Hence we have

$$\left(\sum_{\lambda \in K} \alpha(x + \lambda y) - \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y) \right) \sum_{\mu \in K} f(\mu z) = 0, \quad x, y, z \in G,$$

and we obtain that α satisfies Eq. (3). In view of Corollary 1 we get the form of α .

Now we notice that

$$\begin{aligned} |K| \sum_{\lambda \in K} f(\lambda x) + \sum_{\lambda \in K} \alpha(\lambda x) \sum_{\mu \in K} f(\mu y) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(\lambda x + \mu y) \\ &= \sum_{\mu \in K} \sum_{\lambda \in K} f(\mu y + \lambda x) = |K| \sum_{\mu \in K} f(\mu y) + \sum_{\mu \in K} \alpha(\mu y) \sum_{\lambda \in K} f(\lambda x), \quad x, y \in G. \end{aligned}$$

Hence

$$\left(|K| - \sum_{\lambda \in K} \alpha(\lambda x)\right) \sum_{\mu \in K} f(\mu y) = \left(|K| - \sum_{\mu \in K} \alpha(\mu y)\right) \sum_{\lambda \in K} f(\lambda x), \quad x, y \in G.$$

which gives us Eq. (15) for some $A_0 \in X$. Now, we can write Eq. (4) in the form

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|f(x) + \alpha(x) \left(|K| - \sum_{\lambda \in K} \alpha(\lambda y)\right) A_0, \quad x, y \in G.$$

Let $q: G \rightarrow X$ be given by the formula

$$q(x) = f(x) + \alpha(x)A_0, \quad x \in G.$$

Then from equalities (3), (4), (15) we have

$$\begin{aligned} \sum_{\lambda \in K} q(x + \lambda y) &= \sum_{\lambda \in K} f(x + \lambda y) + \sum_{\lambda \in K} \alpha(x + \lambda y)A_0 \\ &= |K|f(x) + \alpha(x) \sum_{\lambda \in K} f(\lambda y) + \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y)A_0 \\ &= |K|f(x) + |K|\alpha(x)A_0 = |K|q(x), \quad x, y \in G. \end{aligned}$$

In view of Theorem 1 there exist $c \in X$, k -additive, symmetric mappings $A_k: G^k \rightarrow X, k \in \{1, \dots, |K| - 1\}$ such that

$$\begin{aligned} q(x) &= c + \sum_{i=1}^{|K|-1} A_i(x, \dots, x), \quad x \in G, \\ \sum_{\mu \in K} A_k(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad 1 \leq i \leq k < |K|. \end{aligned}$$

Since

$$c = q(0) = f(0) + \alpha(0)A_0 = A_0,$$

we have

$$f(x) = A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \alpha(x)A_0, \quad x \in G.$$

Now we assume that f satisfies conditions (5)–(6) and α satisfies conditions (7)–(9) in the complex case or (10)–(14) in the real case. In view of Theorem 1 a function $f + \alpha A_0$ satisfies Eq. (1) and in view of Corollary 1 α satisfies Eq. (3). We have

$$\begin{aligned} \sum_{\lambda \in K} f(x + \lambda y) &= \sum_{\lambda \in K} (f(x + \lambda y) + \alpha(x + \lambda y)A_0) - \sum_{\lambda \in K} \alpha(x + \lambda y)A_0 \\ &= |K|f(x) + |K|\alpha(x)A_0 - \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y)A_0, \quad x, y \in G. \end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{\lambda \in K} f(\lambda y) &= |K|f(0) + |K|\alpha(0)A_0 - \alpha(0) \sum_{\lambda \in K} \alpha(\lambda y)A_0 \\ &= |K|A_0 - \sum_{\lambda \in K} \alpha(\lambda y)A_0, \quad x, y \in G\end{aligned}$$

and

$$\begin{aligned}\sum_{\lambda \in K} f(x + \lambda y) &= |K|f(x) + |K|\alpha(x)A_0 - \alpha(x) \sum_{\lambda \in K} \alpha(\lambda y)A_0 \\ &= |K|f(x) + \alpha(x) \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G,\end{aligned}$$

which ends the proof. \square

Remark 1. Let $f: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}, \alpha \neq 0$ or $\sum_{\lambda \in K} f \circ \lambda = 0$. Then they satisfy Eq. (15) if and only if f satisfies Eq. (1). Hence, in view of Theorem 1, we know the form of f .

Remark 2. Let $f: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}, \sum_{\lambda \in K} \alpha \circ \lambda = |K|$. If they satisfy Eq. (15) then α satisfies Eq. (1) and we know its form. At the present moment we don't know the form of f .

Now we can prove the main theorem of this paper which is a pexiderized version of Theorem 3.

Theorem 4. Let functions $f, g, h: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}$ be such that $\sum_{\lambda \in K} f \circ \lambda \neq \text{const}$, $\sum_{\lambda \in K} \alpha \circ \lambda \neq \text{const}$, $\alpha(0) \neq 0$. They satisfy the equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + \alpha(x)h(y), \quad x, y \in G, \quad (16)$$

if and only if there exist a homomorphism $m: G \rightarrow \mathbb{C}^*, A, B, A_0 \in X$, k -additive, symmetric mappings $A_k: G^k \rightarrow X, k \in \{1, \dots, |K| - 1\}$ such that

$$f(x) = A + A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \frac{\alpha(x)}{\alpha(0)} A_0, \quad x \in G, \quad (17)$$

$$g(x) = A + A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \frac{\alpha(x)}{\alpha(0)} [A + A_0 - B], \quad x \in G, \quad (18)$$

$$h(x) = \frac{1}{\alpha(0)} \left[\left(|K| - \sum_{\lambda \in K} \frac{\alpha(\lambda x)}{\alpha(0)} \right) A_0 + |K|(A - B) \right], \quad x \in G, \quad (19)$$

$$\sum_{\mu \in K} A_k(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad 1 \leq i \leq k < |K|, \quad (20)$$

and

- (i) if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k -additive, symmetric mappings $a_k^\lambda: G^k \rightarrow \mathbb{C}$, $k \in \{1, \dots, |K_0| - 1\}$, $\lambda \in K_1$ such that

$$\alpha(x) = \alpha(0) \sum_{\lambda \in K_1} m(\lambda x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right], \quad x \in G, \quad (21)$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (22)$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \quad (23)$$

- (ii) if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k -additive, symmetric mappings $a_k^\lambda, b_k^\lambda: G^k \rightarrow \mathbb{R}$, $k \in \{1, \dots, |K_0| - 1\}$, $\lambda \in K_1$ such that

$$\alpha(x) = \alpha(0) \sum_{\lambda \in K_1} \left(\operatorname{Re}(m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right] \right. \quad (24)$$

$$\left. - \operatorname{Im}(m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \dots, x) \right] \right), \quad x \in G, \quad (25)$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (26)$$

$$\sum_{\mu \in K_0} b_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \quad (27)$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \quad (28)$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, K_1 is the set of representatives of cosets of the quotient group K/K_0 .

Proof. Putting $x = 0$ in (16) we have

$$\sum_{\lambda \in K} f(\lambda y) = |K|g(0) + \alpha(0)h(y), \quad y \in G.$$

Putting $y = 0$ in (16) we get

$$|K|f(x) = |K|g(x) + \alpha(x)h(0), \quad x \in G.$$

Hence we get

$$g(x) = f(x) - \frac{\alpha(x)}{|K|}h(0) = f(x) - \frac{\alpha(x)}{\alpha(0)}[f(0) - g(0)], \quad x \in G, \quad (29)$$

$$h(y) = \frac{1}{\alpha(0)} \left[\sum_{\lambda \in K} f(\lambda y) - |K|g(0) \right], \quad y \in G. \quad (30)$$

Let $f_0 = f - f(0)$, $\alpha_0 = \frac{\alpha}{\alpha(0)}$. From the above equalities we obtain

$$\begin{aligned} \sum_{\lambda \in K} f_0(x + \lambda y) &= \sum_{\lambda \in K} f(x + \lambda y) - |K|f(0) = |K|g(x) + \alpha(x)h(y) - |K|f(0) \\ &= |K|f_0(x) - \alpha_0(x)|K|[f(0) - g(0)] + \alpha_0(x) \left[\sum_{\lambda \in K} f(\lambda y) - |K|g(0) \right] \\ &= |K|f_0(x) + \alpha_0(x) \sum_{\lambda \in K} f_0(\lambda y), \quad x, y \in G. \end{aligned}$$

In view of Theorem 3 there exist a homomorphism $m: G \rightarrow \mathbb{C}^*$, $A_0 \in X$, k -additive, symmetric mappings $A_k: G^k \rightarrow X$, $k \in \{1, \dots, |K| - 1\}$ such that

$$\begin{aligned} f_0(x) &= A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \alpha_0(x)A_0, \quad x \in G, \\ \sum_{\mu \in K} A_k(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad 1 \leq i \leq k < |K|, \end{aligned}$$

and

- (i) if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k -additive, symmetric mappings $a_k^\lambda: G^k \rightarrow \mathbb{C}$, $k \in \{1, \dots, |K_0| - 1\}$, $\lambda \in K_1$ such that

$$\begin{aligned} \alpha_0(x) &= \sum_{\lambda \in K_1} m(\lambda x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right], \quad x \in G, \\ \sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\lambda \in K_1} a_0^\lambda &= 1, \end{aligned}$$

- (ii) if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k -additive, symmetric mappings $a_k^\lambda, b_k^\lambda: G^k \rightarrow \mathbb{R}, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$\begin{aligned} \alpha_0(x) &= \sum_{\lambda \in K_1} \left(\operatorname{Re}(m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right] \right. \\ &\quad \left. - \operatorname{Im}(m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \dots, x) \right] \right), \quad x \in G, \\ \sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\mu \in K_0} b_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) &= 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k < |K_0|, \\ \sum_{\lambda \in K_1} a_0^\lambda &= 1. \end{aligned}$$

Moreover

$$\sum_{\lambda \in K} f_0(\lambda x) = \left(|K| - \sum_{\lambda \in K} \alpha_0(\lambda x) \right) A_0, \quad x \in G.$$

Hence, putting $A := f(0), B := g(0)$ and using equalities (29), (30), we obtain the form of f, g, h and α .

Now we assume that f, g, h satisfy conditions (17)–(20) and α satisfies conditions (21)–(23) in the complex case and (24)–(28) in the real case. In view of Theorem 1 a function $f + \alpha_0 A_0$ satisfies Eq. (1) and in view of Corollary 1 α_0 satisfies Eq. (3). We have

$$\begin{aligned} \sum_{\lambda \in K} f(x + \lambda y) &= \sum_{\lambda \in K} (f(x + \lambda y) + \alpha_0(x + \lambda y) A_0) - \sum_{\lambda \in K} \alpha_0(x + \lambda y) A_0 \\ &= |K| f(x) + |K| \alpha_0(x) A_0 - \alpha_0(x) \sum_{\lambda \in K} \alpha_0(\lambda y) A_0 = |K| f(x) \\ &\quad - |K| \alpha_0(x) [A - B] + \alpha_0(x) \left[|K| (A - B) + |K| A_0 - \sum_{\lambda \in K} \alpha_0(\lambda y) A_0 \right] \\ &= |K| g(x) + \alpha(x) h(y), \quad x, y \in G, \end{aligned}$$

which ends the proof. \square

Remark 3. Let $f, g, h: G \rightarrow X, \alpha: G \rightarrow \mathbb{K}$ satisfy Eq. (16).

- (i) If $\alpha(0) = 0$, then $\sum_{\lambda \in K} f \circ \lambda = \text{const}$.
(ii) If $\sum_{\lambda \in K} f \circ \lambda = \text{const}$, then $\sum_{\lambda \in K} \alpha \circ \lambda = \text{const}$ or $h = \text{const}$ (in this case Eq. (16) becomes Eq. (1) and we know its form). We don't know the form of the solutions in the case when $\sum_{\lambda \in K} \alpha \circ \lambda = \text{const}$.

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